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SLOW MOTIONS IN PROBLEMS OF THE DYNAMICS OF A SOLID WITH A CAVITY FILLED WITH A VISCOUS LIQUID†

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The well-known problem of the motion of a solid with a cavity filled with a viscous liquid is considered. The problem is described by a singularly perturbed system of differential equations, consisting of the equations of motion of the solid and the Navier–Stokes equations. The Reynolds number is the small parameter. It is shown that the method of integral manifolds [1, 2] can be used to investigate this system of equations. It enables the investigation of the complete singularly perturbed system to be replaced by an investigation of a regular abbreviated system of lower dimensions, describing the slow motions of the complete system. The equations of the slow motions are obtained, which turn out to be the equations of motion of the solid with additional terms on the right-hand side which occur as a result of taking into account the effect of the viscous liquid on the motion of the solid.

This problem was considered previously [3] in a similar formulation using the Vishik–Lyusternik method [4], and also in [5] using the method of boundary functions [6]. The method of integral manifolds [7] used below has been employed to separate the motions in the problem of the rotation of a conducting solid in a magnetic field.

1. DESCRIPTION OF THE PROBLEM

The motion of a solid with a cavity D completely filled with a viscous incompressible liquid of density ρ and kinematic viscosity ν, in a potential field of mass forces with potential $U(r, t)$ is described by the following system of equations in a system of coordinates x_1, x_2, x_3 rigidly connected with the body [3]

$$
\begin{aligned}
& dz/dt = k(z,\omega) \\
& I d\omega/dt + \rho(d/dt)\int_D (r\times u)dr = M(z,\omega) - \omega\times I\omega - \omega\times\rho\int_D (r\times u)dr \\
& \partial u/\partial t + \partial(\omega\times r)/\partial t = \nu\Delta u - [\nabla q + (u\nabla)u + 2(\omega\times u)] \\
& \operatorname{div} u = 0 \quad (r\in D),\quad u|_{\partial D} = 0,\quad q = p/\rho + U - (\omega\times r)^2/2 - r d\upsilon/dt
\end{aligned}
\tag{1.1}
$$

Here $z = z(t)$ is a vector function, whose components are the kinematic parameters characterizing the orientation of the solid (for example, the Euler angles, or the direction cosines), and also (in the case of an unclamped body) the coordinates and velocity of the centre of mass of the system, $\omega = \omega(t)$ is the vector of the absolute angular velocity of the solid, $u = u(r, t)$ is

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the vector of the relative velocity of the particles of liquid, $p=p(r,\ t)$ is the pressure in the liquid, $\upsilon=\upsilon(t)$ is the absolute velocity of the centre of mass of the system, r is the radius-vector of a given point in the coupled system of coordinates, and t is the time. Further, $I=I_0+I_1$, where I_0 is the inertia tensor of the body, I_1 is the inertia tensor of the liquid, "consolidated" in the body, and $M=M(z,\ \omega)$ is the moment of the external forces.

Suppose t_0 is a characteristic time in the motion of the solid with respect to the centre of mass and l_0 is the characteristic linear dimension of the region D. We will investigate system (1.1) with the condition that the Reynolds number $R=l_0^2t_0^{-1}\nu^{-1}$ is small (equal to the ratio of the characteristic time $l_0^2\nu^{-1}$, during which the viscosity considerably changes the flow in the cavity, to the characteristic time t_0 of relative motion of the body). Without loss of generality we will assume that $l_0=1$ and $t_0=1$. Then, with the above assumptions the parameter $\mu\equiv\nu^{-1}$ will be small and system (1.1) will be singularly perturbed.

2. REFORMULATION OF THE PROBLEM

We will write system (1.1) in the form of the following system of equations

$$\dot{z}=k(z,\omega),\quad a_{11}\dot{\omega}+a_{12}\dot{u}=f(z,\omega,u) \tag{2.1}$$

$$\mu[a_{21}\dot{\omega}+a_{22}\dot{u}]=Au-\mu\nabla q+\mu g(z,\omega,u) \tag{2.2}$$

Here and henceforth the dot denotes differentiation with respect to time, the variables $(z,\ \omega,\ u)$ $\in\mathbb{Z}\times\mathbb{R}^3\times\mathbb{L}^2(D)$, where $\mathbb{Z}$ is a finite-dimensional real space of appropriate dimensions, while $\mathbb{L}$ is the space of quadratically summed vectors defined in the region D. The operator

$$\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}:\mathbb{R}^3\times\mathbb{L}^2(D)\to\mathbb{R}^3\times\mathbb{L}^2(D)$$

is defined by the equations

$$a_{11}\omega=I\omega,\quad a_{12}u=\rho\int_D(r\times u)dr,\quad a_{21}\omega=\omega\times r,\quad a_{22}u=u \tag{2.3}$$

and is linear and bounded. The operator $A:L^2(D)\to\mathbb{L}^2(D):u\to\Delta u$ has the region of definition

$$D(A)=\{u\in\mathbb{W}_2^2(D)\mid \operatorname{div}u=0,\quad u|_{\partial D}=0\}$$

where $\mathbb{W}_2^2(D)$ is a Sobolev space. Finally, the functions f and g are defined by the equations

$$f(z,\omega,u)=f_0(z,\omega)-\omega\times a_{12}u,\quad f_0(z,\omega)=M(z,\omega)-\omega\times I\omega \tag{2.4}$$

$$g(z,\omega,u)=-2(\omega\times u)-(u\nabla)u$$

and are continuous.

We have the following expansion of $\mathbb{L}^2(D)$ in an orthogonal direct sum [8]

$$\mathbb{L}^2(D)=\mathbb{S}_n(D)\oplus\mathbb{G}(D)$$

where the space $\mathbb{S}_n(D)$ is obtained by closure with respect to the norm of the space $\mathbb{L}^2(D)$ of the set of continuous vectors solenoidal in D, having a zero normal component on ∂D, while the space $\mathbb{G}(D)$ is obtained by closure with respect to the norm of the space $\mathbb{L}^2(D)$ of the set of gradients of the functions continuous in D. Suppose $\Pi:\mathbb{L}^2(D)\to\mathbb{L}^2(D)$ is the orthogonal projector onto the subspace $\mathbb{S}_nD$[9]. By projecting Eq. (2.2) onto $\mathbb{S}_nD$ we obtain the equation

$$\mu[\Pi a_{21}\dot{\omega} + a_{22}\dot{u}] = \Pi A u + \mu \Pi g(z, \omega, u) \tag{2.5}$$

The system of equations (2.1), (2.5) is closed with respect to the variables z, ω, u. The operator $\Pi A : \mathbb{S}_n(D) \to \mathbb{S}_n(D)$ is densely defined and, as was shown in [10], is self-conjugate and negative-definite. Note that, since the region D is unbounded, it therefore follows that the operator ΠA has a real discrete spectrum , which lies entirely in the left half-plane $\mathbb{C}^-$, and a system of eigenvectors complete in $\mathbb{S}_n(D)$. System (2.1), (2.5) will be the subject of further investigation.

3. CONVERSION OF THE SYSTEM

We will convert system (2.1), (2.5) to standard form. To do this we introduce the operator

$$L = \begin{Vmatrix} a_{11} & a_{12} \\ \Pi a_{21} & a_{22} \end{Vmatrix} : \mathbb{R}^3 \times \mathbb{S}_n(D) \to \mathbb{R}^3 \times \mathbb{S}_n(D)$$

and rewrite system (2.1), (2.5) in the form

$$\begin{aligned} &\dot{z} = k(z, \omega) \\ &\mu L \frac{d}{dt} \begin{Vmatrix} \omega \\ u \end{Vmatrix} = \begin{Vmatrix} 0 & 0 \\ 0 & \Pi A \end{Vmatrix} \begin{Vmatrix} \omega \\ u \end{Vmatrix} + \mu \begin{Vmatrix} f(z, \omega, u) \\ \Pi g(z, \omega, u) \end{Vmatrix} \end{aligned} \tag{3.1}$$

We will show that the operator L is inverse. First we note that

$$(a_{22} - \Pi a_{21} a_{11}^{-1} a_{12}) u = u - \Pi(\rho I^{-1} \int\limits_D (r \times u) dr \times r) \equiv (E - \Pi B) u$$

The last equality is used to determine the operator $B : \mathbb{L}^2(D) \to \mathbb{L}^2(D)$. It was shown in [5] that $\| B \| < 1$. Since Π is the projector, it follows that $\| \Pi B \| < 1$. Hence, the operator $(E - \Pi B)$ is inverse in $\mathbb{L}^2(D)$ and consequently in $\mathbb{S}_n(D)$. Since the operators a_{11} and a_{22} are obviously inverse, the invertibility of L therefore follows.

We now obtain the explicit form of the operator

$$L^{-1} \equiv \begin{Vmatrix} b_{11} & b_{12} \\ \Pi b_{21} & b_{22} \end{Vmatrix} : \mathbb{R}^3 \times \mathbb{S}_n(D) \to \mathbb{R}^3 \times \mathbb{S}_n(D)$$

We have

$$\begin{aligned} b_{11} &= (a_{11} - a_{12} a_{22}^{-1} \Pi a_{21})^{-1} = J^{-1}, \quad b_{12} = -J^{-1} a_{12} a_{22}^{-1} \\ b_{21} &= -(a_{22} - \Pi a_{21} a_{11}^{-1} a_{12})^{-1} \Pi a_{21} a_{11}^{-1} = -(E - \Pi B)^{-1} \Pi a_{21} I^{-1} \\ b_{22} &= (E - \Pi B)^{-1} \end{aligned} \tag{3.2}$$

(the first equality serves to determine the operator $J : \mathbb{R}^3 \to \mathbb{R}^3$).

We note for later that

$$b_{22}^{-1} b_{21} = -\Pi a_{21} I^{-1}, \quad b_{12} b_{22}^{-1} = -I^{-1} a_{12}, \quad b_{11} - b_{12} b_{22}^{-1} b_{21} = I^{-1} \tag{3.3}$$

We can now write system (3.1) in the standard form

$$\begin{aligned} &\dot{z} = k(z, \omega), \quad \mu\dot{\omega} = \xi_1, \quad \mu\dot{u} = \xi_2 \\ &\xi_k = b_{k2} \Pi A u + \mu[b_{k1} f(z, \omega, u) + b_{k2} \Pi g(z, \omega, u)], \quad k = 1, 2 \end{aligned} \tag{3.4}$$

4. SCHEME FOR CONSTRUCTING THE INTEGRAL MANIFOLD AND EQUATION OF SLOW MOTIONS

Suppose the singularly perturbed system

$$\dot{x} = F(x, y, \mu), \quad \mu\dot{y} = G(x, y, \mu) \tag{4.1}$$

satisfies the following two conditions
1. the generating equation $G(x, y, 0) = 0$ has an isolated solution $y = h_0(x)$;
2. for the isolated solution $y = h_0(x)$ for each x we have the following condition of stability

$$\operatorname{spec}\left(\frac{\partial}{\partial y} G(x, h_0(x), 0)\right) \subset \mathbb{C}^- \tag{4.2}$$

Then [1, 2] system (4.1) has an invariant manifold $y = h(x, \mu)$ lying in a certain neighbourhood of the surface $y = h_0(x)$. and which satisfies the condition of stability and the reduction principle.

The condition of stability. We will represent the solutions of system (4.1), which begin in the region of the manifold $y = h(x, \mu)$, in the form of the sum of a certain solution which lies on the manifold $y = h(x, \mu)$, and a small exponentially decreasing correction. Motion along the manifold $y = h(x, \mu)$ occurs in accordance with the equation

$$\dot{x} = F(x, h(x, \mu), \mu), \quad y = h(x, \mu) \tag{4.3}$$

which is called the equation of slow motion.

The reduction principle [11]. The problems of stability are equivalent for Eqs (4.1) and (4.3). If, in particular, $F(0, 0, \mu) = 0$, $G(0, 0, \mu) = 0$, then $h(0, \mu) = 0$ and the zeroth solution of Eqs (4.1) is stable (asymptotically stable, unstable) if and only if the zeroth solution of Eq. (4.3) possesses a similar property.

The invariant manifold $y = h(x, \mu)$ is called the integral manifold (slow motions) of system (4.1). These properties of the integral manifold enable us to reduce the investigation of the initial system (4.1) to an investigation of system (4.3).

The function $h(x, \mu)$ can be obtained in the form of the asymptotic expansion from the equation

$$\mu \frac{\partial h}{\partial x} F(x, h, \mu) = G(x, h, \mu) \tag{4.4}$$

5. THE EQUATION OF SLOW MOTIONS

If we replace u by μu in (3.4), we obtain a system in which the equations for z and ω are regular, while the equation for u is singularly perturbed. Putting $\mu = 0$ in this equation for u, we obtain the generating equation

$$b_{22}\Pi(Au + g(z, \omega, 0)) + b_{21} f(z, \omega, 0) = 0 \tag{5.1}$$

Since the operator ΠA is inverse, this equation has a unique solution. Using relations (2.4) and (3.3) we can represent this solution in the form

$$u = h_0(z, \omega) = (\Pi A)^{-1} \Pi a_{21} I^{-1} f_0(z, \omega) \tag{5.2}$$

Condition (4.2) in the case considered should have the form

$$\operatorname{spec}(b_{22}\Pi A) \subset \mathbb{C}^-$$

Since the operator b_{22} is inverse, the spectrum of the operator $b_{22}\Pi A$ is identical with the spectrum of the operator ΠA, and consequently lies in the left half-plane $\mathbb{C}^-$. Hence, condition (4.2) is satisfied.

Hence, system (3.4) has an integral manifold of slow motions which can be obtained in the form $u = \mu h(z, \omega, \mu)$ from the equation

$$\begin{aligned} b_{22}\Pi A h = {} & -b_{21}f_0 + \mu[2b_{22}\Pi(\omega\times h) + b_{21}(\omega\times a_{12}h) + \\ & +\left(E+\mu\frac{\partial h}{\partial\omega}I^{-1}a_{12}\right)^{-1}\left(\frac{\partial h}{\partial z}k+\frac{\partial h}{\partial\omega}I^{-1}f_0\right)\Bigg] + \\ & +\mu^2\left[b_{22}(h\nabla)h-\left(E+\mu\frac{\partial h}{\partial\omega}I^{-1}a_{12}\right)^{-1}\frac{\partial h}{\partial\omega}I^{-1}(\omega\times a_{12}h)\right] \end{aligned} \tag{5.3}$$

(we have used representations (2.4)).

Expanding the solution of this equation in the form

$$h(z,\omega,\mu) = h_0(z,\omega) + \mu h_1(z,\omega) + \ldots \tag{5.4}$$

for $h_0(z, \omega)$ we obtain representation (5.2), and for h_1 we obtain the expression

$$\begin{aligned} h_1(z,\omega) = {} & (\Pi A)^{-1}[2\Pi(\omega\times h_0) - \Pi a_{21}I^{-1}(\omega\times a_{12}h_0) + \\ & +(\Pi A)^{-1}\Pi a_{21}I^{-1}\left(\frac{\partial f_0}{\partial z}k+\frac{\partial f_0}{\partial z}I^{-1}f_0\right)\Bigg] \end{aligned}$$

The remaining coefficients $h_k(z, \omega)$ in expansion (5.4) can be obtained in a similar way.

It follows from (4.4) that the equation of slow motions for system (3.4) will have the form

$$\dot z = k(z,\omega), \quad \dot\omega = b_{11}f(z,\omega,\mu h) + b_{12}\Pi g(z,\omega,\mu h) + b_{12}\Pi A h$$

Substituting the expression from (5.3) here instead of $\Pi A h$ we obtain the following equation for ω

$$\begin{aligned} I\dot\omega = {} & f_0 - \mu\left[(\omega\times a_{12}h) + a_{12}\left(E+\mu\frac{\partial h}{\partial\omega}I^{-1}a_{12}\right)^{-1}\left(\frac{\partial h}{\partial z}k+\frac{\partial h}{\partial\omega}I^{-1}f_0\right)\right] + \\ & +\mu^2\left[a_{12}\left(E+\mu\frac{\partial h}{\partial\omega}I^{-1}a_{12}\right)^{-1}\frac{\partial h}{\partial\omega}I^{-1}(\omega\times a_{12}h)\right] \end{aligned} \tag{5.5}$$

Substituting expansion (5.4) here instead of h, expanding in powers of μ, introducing the notation

$$P = -\rho^{-1}a_{12}(\Pi A)^{-1}\Pi a_{21} \tag{5.6}$$

and using representation (5.2) we obtain the equation of slow motions of system (3.4) and, of course, of system (1.1) also in the form

$$\dot z = k(z,\omega) \tag{5.7}$$

$$I\dot{\omega} = f_0 + \mu\rho\left[\omega\times PI^{-1}f_0 + PI^{-1}\left(\frac{\partial f_0}{\partial z}k + \frac{\partial f_0}{\partial \omega}I^{-1}f_0\right)\right] + O(\mu^2)$$

Equations (5.7) are identical with those obtained previously in [3, 5]. Here the equation of slow motions (5.7) is written exactly to terms $O(\mu^2)$, but using the equation for the integral manifold (5.3) and Eq. (5.5) for ω we can easily write the equation of slow motions with any degree of accuracy.

6. PROPERTIES OF THE OPERATOR P

We will prove that the operator P, defined by (5.6), is self-conjugate and positive-definite. Hence, it will follow, in particular, that in a certain system of coordinates the matrix of the operator P is diagonal and has positive values on the principal diagonal.

We first of all note that the operator $\Pi a_{21}:\mathbb{R}^3\to\mathbb{S}_n(D)$ can be represented in the form

$$(\Pi a_{21})\omega = \omega\times r + \Psi(r)\omega$$

where $\Psi(r) = \mathrm{mat}[\nabla\psi_1,\ \nabla\psi_2,\ \nabla\psi_3]$ is a matrix (in the system of coordinates the columns of which will be the vectors $\nabla\psi_k$, while the functions ψ_k are the solutions of the Stokes–Zhukovskii problem [12]

$$\Delta\psi_k = 0,\quad \partial\psi_k/\partial n + (e_k\times r, n)|_{\partial D} = 0\quad (k=1,2,3)$$

where e_k are the unit vectors of the system of coordinates x_1, x_2, x_3. The operator $(\Pi A)^{-1}$: $\mathbb{S}_n(D)\to\mathbb{S}_n(D)$ is bijective, self-conjugate and negative-definite [10], while $(\Pi A)^{-1}u = w$ in the case when w is the solution of the problem

$$-\mathrm{rot}\,\mathrm{rot}\,w = u,\quad \mathrm{div}\,w = 0\quad (r\in D),\quad w|_{\partial D} = 0$$

We finally recall that the operator $a_{12}:\mathbb{S}_n(D)\to\mathbb{R}^3$ is defined by (2.3).

Further, for all $u\in\mathbb{S}_n(D)$ and $\omega\in\mathbb{R}^3$ we have

$$(\rho^{-1}a_{12}u,\omega) = \int_D (r\times u,\omega)dr = \int_D(\omega\times r, u)dr =$$
$$= \int_D(\Pi(\omega\times r), u)dr - \int_D(\Psi(r)\omega, u)dr = (u, \Pi a_{21}\omega)$$

This means that $(\rho^{-1}a_{12})^* = \Pi a_{21}$, where the asterisk denotes a change to the conjugate operator. Hence, from the fact that the operator ΠA is a self-conjugate it follows that the operator P is self-conjugate.

Further, for all $\omega\in\mathbb{R}^3$, $\omega\neq 0$ we have

$$= \|\mathrm{rot}\,\Omega\|^2 + \int_{\partial D}(\Omega, n\times\mathrm{rot}\,\Omega)ds = \|\mathrm{rot}\,\Omega\|^2 > 0\quad (\Omega = (\Pi A)^{-1}\Pi a_{21}\omega)$$

This means that the operator P is positive-definite.

Note that in the conversions in Section 6 the same symbol (. , .) has been used to denote the scalar product both in the space $\mathbb{R}^3$ and in the space $\mathbb{L}^2(D)$.

REFERENCES

1. MITROPOL'SKII Yu. A. and LYKOVA O. B., *Integral Manifolds in Non-linear Mechanics.* Nauka, Moscow, 1973.
2. STRYGIN V. V. and SOBOLEV V. A., *Separation of Motions by the Method of Integral Manifolds.* Nauka, Moscow, 1988.
3. CHERNOUS'KO F. L., The motion of a solid with cavities filled with a viscous liquid for small Reynolds numbers. *Zh. Vychisl. Mat. Mat. Fiz.* **5**, 6, 1049–1070, 1965.
4. VISHIK M. I. and LYUSTERNIK L. A., Regular degeneration and the boundary layer for linear differential equations with a small parameter. *Usp. Mat. Nauk* **12**, 5, 3–122, 1957.
5. KOBRIN A. I., The problem of the motion of a body with a cavity filled with a viscous liquid about the centre of mass in a potential field of mass forces. *Prikl. Mat. Mekh.* **33**, 3, 431–440, 1969.
6. VASIL'YEVA A. B. and BUTUZOV V. F., *Asymptotic Expansions of the Solutions of Singularly Perturbed Equations.* Nauka, Moscow, 1973.
7. BOGATYREV S. V. and SOBOLEV V. A., Separation of fast and slow motions in problems of the dynamics of systems of solids and gyroscopes. *Prikl. Mat. Mekh.* **52**, 1, 47–54, 1988.
8. BYKHOVSKII E. B. and SMIRNOV N. V., The orthogonal expansion of the space of vector functions, quadratically symmetrical with respect to a specified region and the operators of vector analysis. *Trudy Mat. Inst. im V. A. Steklova* **59**, 5–36, 1960.
9. KREIN S. G., The functional properties of the operators of vector analysis and hydrodynamics. *Dokl. Akad. Nauk SSSR* **93**, 6, 969–972, 1953.
10. YUDOVICH V. I., *The Method of Linearization in the Hydrodynamic Theory of Stability.* Izd. Rostov. Univ., Rostov, 1984.
11. PLISS V. A., The reduction principle in the theory of the stability of motion. *Izv. Akad. Nauk SSSR, Ser. Mat.* **28**, 6, 1297–1324, 1964.
12. MOISEYEV N. N. and RUMYANTSEV V. V., *The Dynamics of a Body with Cavities Containing Liquid.* Nauka, Moscow, 1965.

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